Analysis SEP 2026: The Arzelà-Ascoli Theorem

1 Warm-up

1. Read §3.3 of Andreas Seeger and Joris Roos' math 522 notes.
2. What does it mean for a family of sets to be "totally bounded"?
3. Give the precise statement of the Arzelà-Ascoli theorem.

2 Exercises

Problem 1.24.1 Let $K_n : [a,b] \times [a,b] \to \mathbb{R}$ be a sequence of differentiable functions such that

$$\max_{[a,b]^2} |K_n(x,t)| \le 1, \quad \max_{[a,b]^2} \left| \frac{\partial K_n}{\partial t}(x,t) \right| \le 1, \quad \max_{[a,b]^2} \left| \frac{\partial K_n}{\partial x}(x,t) \right| \le 1.$$

Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. FOr $f \in C[a,b]$, define

$$A_n f(x) = \int_a^b K_n(x,t) f(t)\, dt.$$

(i) Prove that $\{A_n f : \max_{[a,b]} |f(x)| \le 1\}$ is a totally bounded subset of $C[a,b]$.

(ii) If in (i) we drop the assumption $\max_{[a,b]^2} \left| \frac{\partial K_n}{\partial x}(x,t) \right| \le 1$, and keep the other assumptions, does $\{A_n f : \max_{[a,b]} |f(x)| \le 1\}$ have to be a totally bounded subset of $C[a,b]$?

(iii) If in (i) we drop the assumption $\max_{[a,b]^2} \left| \frac{\partial K_n}{\partial t}(x,t) \right| \le 1$, and keep the other assumptions, does $\{A_n f : \max_{[a,b]} |f(x)| \le 1\}$ have to be a totally bounded subset of $C[a,b]$?

Problem 8.22.1 Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions, each of which has continuous first and second derivatives on $(0,1)$. Prove: If

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \text{for all} \quad x \in [0,1]$$

and

$$\sup_{n \ge 1} \max_{0<x<1} |f''(x)| < +\infty,$$

then f' exists and is continuous on $(0,1)$.

Problem 8.23.2 Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. Consider the space $C[a,b]$ of continuous functions on $[a,b]$ with the sup-norm. For $f \in C[a,b]$, define

$$S_K f(x) = \int_a^b K(x,t) f(t)\, dt.$$

(i) Is $\{S_k f : \max_{[a,b]} |f(x)| \leq 1\}$ necessarily a totally bounded subset of $C[a,b]$?

(ii) Let f_n be a sequence of continuous functions on $[a,b]$ satisfying

$$\sup_n \sup_{x \in [a,b]} |f_n(x)| \leq 1.$$

Does the sequence $S_K f_n$ necessarily have a convergent subsequence? Give proof or a counterexample.

(iii) Let K_n be a sequence of continuous functions on $[a,b] \times [a,b]$ and assume that

$$\sup_n \max\{|K_n(x,y)| : (x,y) \in [a,b]^2\} \leq 1.$$

Let $f \in C[a,b]$. Does the sequence $S_{K_n} f$ necessarily have a convergent subsequence in $C[a,b]$? (Give a proof or counterexample).

Problem 1.14.2 Let B denote the unit ball in $C^2([0,1])$. Consider B as a subset of $C^1([0,1])$. Prove that the closure of B is compact in $C^1([0,1])$.

Problem 1.23.3 Suppose S is the set of real-valued functions continuous on $[0,1]$ that satisfy two conditions:

$$\left| \int_0^1 g\, dx \right| \leq 1$$

and

$$|g(x) - g(y)| \leq |x - y|^{1/2}$$

for each $x, y \in [0,1]$. Consider the functional

$$F(g) = \int_0^1 (1 - 5x^2) g^{10}\, dx.$$

Is F bounded on S? Does it achieve its maximum and minimum on S?